Infrared finite cross sections at NNLO

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I discuss methods for the cancellation of infrared divergences at NNLO.

1. Introduction

In recent years there has been significant progress in the calculation of two-loop amplitudes [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. These amplitudes are needed for fully differential next-to-next-to-leading order (NNLO) calculations to improve the accuracy of theoretical predictions relevant to high-energy collider experiments. At the next-to-next-to-leading order level the ingredients for the third order term in the perrturbative expansion for quantities depending on n resolved "hard" partons are the already mentionend n-parton two-loop amplitudes, the $(n+1)$-parton one-loop amplitudes and the $(n+2)$ Born amplitudes. Taken separately, each one of these contributions is infrared divergent. Only the sum of all contributions is infrared finite. Here I review the state of the art for the cancellation of infrared divergences between these different contributions at NNLO.

Infrared divergences occur already at next-to-leading order. At NLO real and virtual corrections contribute. The virtual corrections contain the loop integrals and can have, in addition to ultraviolet divergences, infrared divergences. If loop amplitudes are calculated in dimensional regularisation, the IR divergences manifest themselves as explicit poles in the dimensional regularisation parameter $\varepsilon = 2 - D/2$. These poles cancel with similar poles arising from amplitudes with additional partons but less internal loops, when integrated over phase space regions where two (or more) partons become "close" to each other. In general, the Kinoshita-Lee-Nauenberg theorem [14, 15] guarantees that any infrared-safe observable, when summed over all states degenerate according to some resolution criteria, will be finite. However, the cancellation occurs only after the integration

over the unresolved phase space has been performed and prevents thus a naive Monte Carlo approach for a fully exclusive calculation. It is therefore necessary to cancel first analytically all infrared divergences and to use Monte Carlo methods only after this step has been performed. At NLO, general methods to circumvent this problem are known. This is possible due to the universality of the singular behaviour of the amplitudes in soft and collinear limits. Examples are the phase-space slicing method [16, 17, 18] and the subtraction method [19, 20, 21, 22, 23].

I briefly review the subtraction method here. Within the subtraction method one subtracts a suitable approximation term $d\sigma^A$ from the real corrections $d\sigma^R$. This approximation term must have the same singularity structure as the real corrections. If in addition the approximation term is simple enough, such that it can be integrated analytically over a one-parton subspace, then the result can be added back to the virtual corrections $d\sigma^V$.

$$\sigma^{NLO} = \int_{n+1} \left(d\sigma^R - d\sigma^A \right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A \right).$$

Since by definition $d\sigma^A$ has the same singular behaviour as $d\sigma^R$, $d\sigma^A$ acts as a local counter-term and the combination $(d\sigma^R - d\sigma^A)$ is integrable and can be evaluated numerically. Secondly, the analytic integration of $d\sigma^A$ over the one-parton subspace will yield the explicit poles in ε needed to cancel the corresponding poles in $d\sigma^V$. The simplest example are the NLO corrections to $\gamma^* \to 2$ jets. The real corrections are given by the matrix element for $\gamma^* \to q(p_1)g(p_2)\bar{q}(p_3)$ and read, up to colour and coupling factors

$$|\mathcal{A}_3|^2 = 8(1-\varepsilon)$$

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$$\left[\frac{2}{x_1 x_2} - \frac{2}{x_1} - \frac{2}{x_2} + (1-\varepsilon)\frac{x_2}{x_1} + (1-\varepsilon)\frac{x_1}{x_2} - 2\varepsilon\right],$$

where $x_1 = s_{12}/s_{123}$ and $x_2 = s_{23}/s_{123}$. Singularities occur at the boundaries of the integration region at $x_1 = 0$ and $x_2 = 0$. The approximation term can be taken as a sum of two (dipole) subtraction terms:

$$d\sigma^A = \left|\mathcal{A}_2(p_1', p_3')\right|^2 \frac{1}{s_{123}}\left[\frac{2}{x_1(x_1+x_2)} - \frac{2}{x_1}\right.$$
$$+ (1-\varepsilon)\frac{x_2}{x_1}\right] + \left|\mathcal{A}_2(p_1'', p_3'')\right|^2 \frac{1}{s_{123}}\left[\frac{2}{x_2(x_1+x_2)}\right.$$
$$\left. - \frac{2}{x_2} + (1-\varepsilon)\frac{x_1}{x_2}\right].$$

The momenta p_1', p_3', p_1'' and p_3'' are linear combinations of the original momenta p_1, p_2 and p_3. The first term is an approximation for $x_1 \to 0$, whereas the second term is an approximation for $x_2 \to 0$. Note that the soft singularity is shared between the two dipole terms and that in general the Born amplitudes $\mathcal{A}_2$ are evaluated with different momenta. The subtraction terms can be derived by working in the axial gauge. In this gauge only diagrams where the emission occurs from external lines are relevant for the subtraction terms. Alternatively, they can be obtained from off-shell currents and antenna factorisation [24, 25, 26, 27].

Once suitable subtraction terms are found, they have to be integrated over the unresolved phase space. Here, one faces integrals with overlapping divergences, as one can already see from our simple example:

$$\int d^3x \delta\left(1 - \sum_{i=1}^{3} x_i\right) x_1^{-\varepsilon} x_2^{-\varepsilon} x_3^{-\varepsilon}$$
$$\left[\frac{2}{x_1(x_1+x_2)} - \frac{2}{x_1} + (1-\varepsilon)\frac{x_2}{x_1}\right] \qquad (1)$$

The term $1/(x_1 + x_2)$ is an overlapping singularity. Sector decomposition [28, 29, 30, 31] is a convenient tool to disentangle overlapping singularities. Other techniques make use of the optical theorem to convert phase space integrals into loop integrals [32]. First applications of these methods to processes like $e^+e^- \to 2$ jets or $pp \to W$ have become available [33, 34, 35, 36, 37].

2. The subtraction method at NNLO

The following terms contribute at NNLO:

$$d\sigma_{n+2}^{(0)} = \left(\mathcal{A}_{n+2}^{(0)\,*}\,\mathcal{A}_{n+2}^{(0)}\right) d\phi_{n+2},$$
$$d\sigma_{n+1}^{(1)} = \left(\mathcal{A}_{n+1}^{(0)\,*}\,\mathcal{A}_{n+1}^{(1)} + \mathcal{A}_{n+1}^{(1)\,*}\,\mathcal{A}_{n+1}^{(0)}\right) d\phi_{n+1},$$
$$d\sigma_n^{(2)} =$$
$$\left(\mathcal{A}_n^{(0)\,*}\,\mathcal{A}_n^{(2)} + \mathcal{A}_n^{(2)\,*}\,\mathcal{A}_n^{(0)} + \mathcal{A}_n^{(1)\,*}\,\mathcal{A}_n^{(1)}\right) d\phi_n,$$

where $\mathcal{A}_n^{(l)}$ denotes an amplitude with n external partons and l loops. $d\phi_n$ is the phase space measure for n partons. We would like to construct a numerical program for an arbitrary infrared safe observable O. Infrared safety implies that whenever a $n+l$ parton configuration $p_1,...,p_{n+l}$ becomes kinematically degenerate with a n parton configuration $p_1',...,p_n'$ we must have

$$O_{n+l}(p_1,...,p_{n+l}) \quad \to \quad O_n(p_1',...,p_n'). \qquad (2)$$

To render the individual contributions finite, one adds and subtracts suitable pieces [38, 39]:

$$\langle O \rangle_n^{NNLO} =$$
$$\int O_{n+2}\, d\sigma_{n+2}^{(0)} - O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_n \circ d\alpha_n^{(0,2)}$$
$$+ \int O_{n+1}\, d\sigma_{n+1}^{(1)} + O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_n \circ d\alpha_n^{(1,1)}$$
$$+ \int O_n\, d\sigma_n^{(2)} + O_n \circ d\alpha_n^{(0,2)} + O_n \circ d\alpha_n^{(1,1)}.$$

Here $d\alpha_{n+1}^{(0,1)}$ is a subtraction term for single unresolved configurations of Born amplitudes. This term is already known from NLO calculations. The term $d\alpha_n^{(0,2)}$ is a subtraction term for double unresolved configurations. Finally, $d\alpha_n^{(1,1)}$ is a subtraction term for single unresolved configurations involving one-loop amplitudes.

To construct these terms the universal factorisation properties of QCD amplitudes in unresolved limits are essential. QCD amplitudes factorise if they are decomposed into primitive amplitudes. Primitive amplitudes are defined by a fixed cyclic ordering of the QCD partons, a definite routing of the external fermion lines through the diagram and the particle content circulating in the loop. One-loop amplitudes factorise in single unresolved limits as [
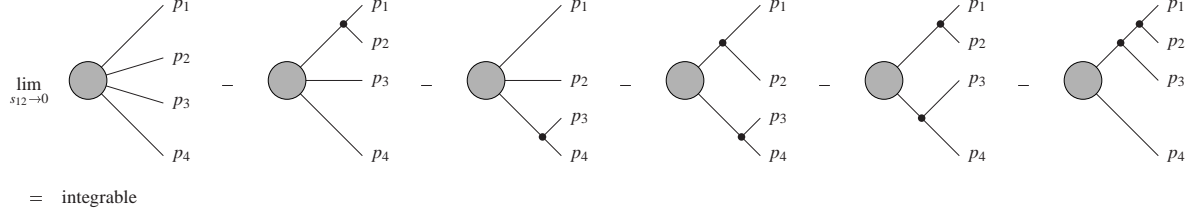
Figure 1. Combinations of subtraction terms, which render the single unresolved limit $s_{12} \to 0$ finite.

40, 41, 42, 43, 44, 45, 26]

$$A_n^{(1)} = \mathrm{Sing}^{(0,1)} \cdot A_{n-1}^{(1)} + \mathrm{Sing}^{(1,1)} \cdot A_{n-1}^{(0)}. \qquad (3)$$

Tree amplitudes factorise in the double unresolved limits as [46, 47, 48, 49, 50, 51, 25]

$$A_n^{(0)} = \mathrm{Sing}^{(0,2)} \cdot A_{n-2}^{(0)}. \qquad (4)$$

To discuss the term $d\alpha_n^{(0,2)}$ let us consider as an example the Born leading-colour contributions to $e^+e^- \to q g g \bar{q}$, which contribute to the NNLO corrections to $e^+e^- \to 2$ jets. The subtraction term has to match all double and single unresolved configurations. It is convenient to construct $d\alpha_n^{(0,2)}$ as a sum over several pieces [38],

$$d\alpha_n^{(0,2)} = \sum_{\text{topologies } T} \mathcal{D}_n^{(0,2)}(T). \qquad (5)$$

Each piece is labelled by a splitting topology. Care has to be taken to disentangle correctly overlapping singularities, such that the integrand is finite in all double and single unresolved limits. Fig. 1 shows a combination which is finite in the single unresolved limit $s_{12} \to 0$. The integration over the double unresolved phase space involves square roots and leads to new types of integrals with half-integer powers. Mapping these integrals to sums, the new types are related to sums of the form [52]

$$\frac{1}{\Gamma\left(\frac{1}{2}\right)} \sum_{n=1}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \frac{x^n}{n^2} = 2 \left[\ln 2 \ln\left(1 + \sqrt{1-x}\right) \right.$$
$$+ \ln 2 \ln\left(1 - \sqrt{1-x}\right) - \mathrm{Li}_{11}\left(-\sqrt{1-x}, 1\right)$$
$$\left. - \mathrm{Li}_{11}\left(\sqrt{1-x}, -1\right) - \mathrm{Li}_2(-1) - (\ln 2)^2 \right],$$

$$\Gamma\left(\frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \frac{x^n}{n^2} = -\mathrm{Li}_{11}\left(\chi, 1\right)$$
$$+ \mathrm{Li}_{11}\left(\chi, -1\right) - \mathrm{Li}_{11}\left(-\chi, 1\right) + \mathrm{Li}_{11}\left(-\chi, -1\right),$$

where $\chi = \sqrt{-x/(1-x)}$.

2.1. One-loop amplitudes with one unresolved parton

Apart from $d\alpha_n^{(0,2)}$ also the term $d\alpha_n^{(1,1)}$, which approximates one-loop amplitudes with one unresolved parton, is needed at NNLO. If we recall the factorisation formula (3), this requires as a new feature the approximation of the one-loop singular function $\mathrm{Sing}^{(1,1)}$. The corresponding subtraction term is proportional to the one-loop $1 \to 2$ splitting function $\mathcal{P}_{(1,0)\, a \to bc}^{(1,1)}$. An example is the leading-colour part for the splitting $q \to qg$ [39]:

$$\mathcal{P}_{(1,0)\, q \to qg, lc, corr}^{(1,1)} = -\frac{11}{6\varepsilon} \mathcal{P}_{q \to qg}^{(0,1)} + S_\varepsilon^{-1} c_\Gamma \left(\frac{-s_{ijk}}{\mu^2} \right)^{-\varepsilon}$$
$$y^{-\varepsilon} \left\{ g_{1,corr}(y,z)\, \mathcal{P}_{q \to qg}^{(0,1)} + f_2 \frac{2}{s_{ijk}} \frac{1}{y} p\!\!\!/_e \right.$$
$$\left. [1 - \rho\varepsilon(1-y)(1-z)] \right\}$$

This term depends on the correlations among the remaining hard partons. If only two hard partons are correlated, g_1 is given by

$$g_{1,intr}(y,z) = -\frac{1}{\varepsilon^2} \left[\Gamma(1+\varepsilon)\Gamma(1-\varepsilon) \left(\frac{z}{1-z} \right)^\varepsilon + 1 \right.$$
$$\left. - (1-y)^\varepsilon z^\varepsilon {}_2F_1\left(\varepsilon, \varepsilon, 1+\varepsilon; (1-y)(1-z)\right) \right].$$

Here, $y = s_{ij}/s_{ijk}$, $z = s_{ik}/(s_{ik}+s_{jk})$ and $f_2 = (1 - \rho\varepsilon)/2/(1-\varepsilon)/(1-2\varepsilon)$. The parameter ρ specifies the variant of dimensional regularisation: $\rho = 1$ in the

conventional or 't Hooft-Veltman scheme and $\rho = 0$ in a four-dimensional scheme. For the integration of the subtraction terms over the unresolved phase space all occuring integrals are reduced to standard integrals of the form

$$\int_0^1 dy\, y^a (1-y)^{1+c+d} \int_0^1 dz\, z^c (1-z)^d \left[1 - z(1-y)\right]^e$$

$$_2F_1\left(\varepsilon, \varepsilon; 1+\varepsilon; (1-y)z\right) =$$

$$\frac{\Gamma(1+a)\Gamma(1+d)\Gamma(2+a+d+e)\Gamma(1+\varepsilon)}{\Gamma(2+a+d)\Gamma(\varepsilon)\Gamma(\varepsilon)}$$

$$\sum_{j=0}^{\infty} \frac{\Gamma(j+\varepsilon)\Gamma(j+\varepsilon)\Gamma(j+1+c)}{\Gamma(j+1)\Gamma(j+1+\varepsilon)\Gamma(j+3+a+c+d+e)}.$$

The result is proportional to hyper-geometric functions $_4F_3$ with unit argument and can be expanded into a Laurent series in ε with the techniques of [53, 54]. For the example discussed above one finds after integration [39]:

$$\mathcal{V}_{(1,0)\,q\to qg,lc,intr}^{(1,1)} = -\frac{1}{4\varepsilon^4} - \frac{31}{12\varepsilon^3} + \left(-\frac{51}{8} - \frac{1}{4}\rho\right.$$

$$\left. + \frac{5}{12}\pi^2 - \frac{11}{6}L\right)\frac{1}{\varepsilon^2} + \left(-\frac{151}{6} - \frac{55}{24}\rho + \frac{145}{72}\pi^2\right.$$

$$\left. + \frac{15}{2}\zeta_3 - \frac{11}{4}L - \frac{11}{12}L^2\right)\frac{1}{\varepsilon} - \frac{1663}{16} - \frac{233}{24}\rho$$

$$+ \frac{107}{16}\pi^2 + \frac{5}{12}\rho\pi^2 + \frac{356}{9}\zeta_3 - \frac{1}{72}\pi^4 - \frac{187}{24}L$$

$$- \frac{11}{12}\rho L + \frac{55}{72}\pi^2 L - \frac{11}{8}L^2 - \frac{11}{36}L^3 + i\pi\left[-\frac{1}{4\varepsilon^3}\right.$$

$$- \frac{3}{4\varepsilon^2} + \left(-\frac{29}{8} - \frac{1}{4}\rho + \frac{\pi^2}{3}\right)\frac{1}{\varepsilon} - \frac{139}{8} - \frac{11}{8}\rho$$

$$\left. + \pi^2 + \frac{15}{2}\zeta_3\right] + O(\varepsilon),$$

where $L = \ln(s_{ijk}/\mu^2)$.

3. Outlook

In this talk I discussed methods for the cancellation of infrared singularities at NNLO. The handling of these divergences is the remaining bottleneck in the construction of fully differential numerical programs at NNLO. With the progress we witnessed in the field in the last years we can expect to obtain numerical results rather soon. An example would be the extension

of exisiting numerical programs for NLO predictions on $e^+e^- \to 4$ jets [55, 56, 57, 58] towards NNLO predictions for $e^+e^- \to 3$ jets.

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